W23. Daniel Sitaru

Prove that if $a, b, c \in (0, 1]$ then:

$$(e^{a^2} - 1)(e^{b^2} - 1)(e^{c^2} - 1) \le (e - 1)^3 a^2 b^2 c^2.$$

Solution by Arkady Alt , San Jose ,California, USA.

Since $e^x \ge 1 + x$ for any real x then in particular for any $x \in [0, 1]$ holds inequality $e^{-x} \ge 1 - x \iff 1 - e^x + xe^x \ge 0$ (with equality iff $x = 0$).

Let $h(x) := \dfrac{e^x - 1}{x}, x \in (0, 1]$. Since $h'(x) = \dfrac{xe^x - e^x + 1}{x^2} > 0$ for any $x \in (0, 1]$ then function $h(x)$ strictly increasing on $(0, 1]$ and, therefore, $h(x) \le h(1) \iff$

$\dfrac{e^x - 1}{x} \le \dfrac{e^1 - 1}{1} = e - 1 \iff e^x - 1 \le (e - 1)x$ with equality only if $x = 1$.

Hence, $(e^{a^2} - 1)(e^{b^2} - 1)(e^{c^2} - 1) \le (e - 1)^3 a^2 b^2 c^2$.